HIGHER DIMENSIONAL SPHERICALLY SYMMETRIC GRAVITATIONAL THEORIES

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Emre Sermutlu
September 1994

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Metin Gürses(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

HIGHER DIMENSIONAL SPHERICALLY SYMMETRIC GRAVITATIONAL THEORIES

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M.S. in Mathematics
Supervisor: Prof. Dr. Metin Gürses
September 1994

We consider all possible theories in spherically symmetric Riemannian geometry in D-dimensions. We find solutions to such theories, in particular black hole solutions of the low energy limit of the string theory in D-dimensions.

Keywords : Gravitation, higher dimensions, black holes, low energy limit, spherical symmetry.

ÖZET

YÜKSEK BOYUTLARDA KÜRESEL SİMETRİK GRAVİTASYON TEORİLERİ

Emre Sermutlu
Matematik Yüksek Lisans
Tez Yöneticisi: Prof. Dr. Metin Gürses
Eylül 1994

D boyutlu küresel simetrik Riemann geometrisinde mümkün olan bütün teorileri ele alarak çözümleri inceledik. Bir özel hal olarak sicim teorisinin düşük enerji limitinde kara delik çözümleri bulduk.

Anahtar Kelimeler : Gravitasyon, yüksek boyutlar, kara delikler, düşük enerji limiti, küresel simetri.

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Chapter 1

Introduction

In the classical relativity, the lagrangian contains only the Ricci scalar. On the other hand we learned from the low energy limit of string theory that the classical lagrangian contains all possible invariants constructed from the curvature tensor and the matter fields. Depending upon the order of the string tension parameter, this lagrangian is an infinite series expansion in these invariants, i.e.

$$L = \sqrt{-g}R + \sum_{n=1}^{\infty} \alpha^n L_n. \tag{1.1}$$

Here α is the inverse of the string tension and L_n are functions containing the invariants up to n^{th} order. There are several examples where the sum is terminated at some value n. For instance when $n = 2$ we have Gauss-Bonnet theory

$$L = \sqrt{-g}R + \alpha(R^{ijkl}R_{ijkl} - 4R^{ij}R_{ij} + R^2). \tag{1.2}$$

For different n we have the Lovelock theorem: In d-dimensions divergence-free second order symmetric tensors constructed from the metric and its first two derivatives are given by

$$A^i_j = \sum_{p=1}^{m-1} a_p \delta^{ih_1 \ldots h_{2p}}_{jj_1 \ldots j_{2p}} R^{j_1 j_2}_{h_1 h_2} R^{j_3 j_4}_{h_3 h_4} \cdots R^{j_{2p-1} j_{2p}}_{h_{2p-1} h_{2p}} + a\delta^i_j, \tag{1.3}$$

where $m = \frac{1}{2}n$ if n is even, $m = \frac{1}{2}(n+1)$ if n is odd.

Boulware and Deser [2] found two solutions for the Einstein plus Gauss-Bonnet lagrangian, one of them asymptotically flat, the other asymptotically anti-de Sitter.

Wiltshire [4] slightly generalized the previous results by including a Maxwell field. Wheeler [5] has considered the most general second-order gravity theory in arbitrary dimensions and analyzed asymptotically flat spherically symmetric static solutions, and cosmological solutions. Whitt [6] extended Wheeler's work to non-static space-times.

The Schwarzschild solution which describes the uncharged black holes in general relativity also describes (to a good approximation) uncharged black holes in string theory. But, this is not the case for charged black holes. The dilaton has a coupling to F^2, so the Reissner-Nordström solution is not even an approximate solution of string theory.

Charged black holes were first analyzed by Gibbons-Maeda [7] and GHS (Garfinkle, Horowitz, Strominger) [8] independently.

Black hole solutions depend on dimension and number of physical parameters. In this work we find D-dimensional solutions with three independent parameters, mass M, electric charge Q, and the dilaton charge Σ. We also extend the Levi-Civita Bertotti-Robinson metrics to D-dimensions and prove that they solve all field equation arising from a variational principle.

Chapter 2

Spherically Symmetric Riemannian Geometry

The metric of a static and spherically symmetric D-dimensional spacetime is given by:

$$ds^2 = -A^2dt^2 + B^2dr^2 + C^2d\Omega^2_{D-2}, \tag{2.1}$$

where A,B,C depend only on r. $d\Omega^2_{D-2}$ is the metric on S_{D-2}. The metric can be rewritten as $g_{ij} = -A^2t_it_j + B^2k_ik_j + C^2h_{ij}$, where $t_i = \delta^t_i$, $k_i = \delta^r_i$, h_{ij} =metric on D-2 sphere for $i,j \geq 2$, $h_{0i} = h_{1i} = 0$.

Christoffel symbols are given by

$$\Gamma^i_{jm} = \frac{1}{2}g^{in}(g_{jn,m} + g_{nm,j} - g_{jm,n}) \tag{2.2}$$

$$\begin{aligned}\Gamma^i_{jm} &= -AA'\,t^i(t_jk_m + t_mk_j) + AA'\,k^it_jt_m + BB'\,k^ik_jk_m - CC'\,k^ih_{jm} \\ &\quad +CC'\,(h^i_jk_m + h^i_mk_j) + \Gamma^i_{(s)jm},\end{aligned} \tag{2.3}$$

where $\Gamma_{(s)}$ is the christoffel symbol on $D-2$ sphere.

The Riemann tensor is given by

$$R^i_{jml} = \Gamma^i_{jl,m} - \Gamma^i_{jm,l} + \Gamma^i_{nm}\Gamma^n_{jl} - \Gamma^i_{nl}\Gamma^n_{jm} \tag{2.4}$$

$$
\begin{aligned}
R_{ijml} &= \left(AA'' - \frac{AA'B'}{B}\right) t_{[i}k_{j]}t_{[m}k_{l]} + \left(CC'' - \frac{CC'B'}{B}\right) k_{[i}h_{j][m}k_{l]} \\
&+ \frac{AA'CC'}{B^2} t_{[m}h_{l][j}t_{i]} - \frac{C^2C'^2}{B^2} h_{i[m}h_{l]j} + C^2 R_{sijml},
\end{aligned} \tag{2.5}
$$

where $R_{sijml} = h_{i[m}h_{l]j}$.
Riemann tensor can be rewritten as

$$
R_{ijml} = g_{jl}S_{im} - g_{jm}S_{il} + g_{im}S_{jl} - g_{il}S_{mj} + \eta_2 H_{ijk\cdots n}H_{ml}^{k\cdots n} \tag{2.6}
$$

where

$$
S_{ij} = \eta_0 M_{ij} + \eta_1 k_i k_j + \frac{1}{2}\eta_3 g_{ij}, \tag{2.7}
$$

$$
M_{ij} = H_{im\cdots n}H_j^{m\cdots n} - \frac{1}{2(D-2)} H^2 g_{ij} \tag{2.8}
$$

which turns out to be

$$
M_{ij} = \frac{(D-3)!}{C^{2(D-3)}}(h_{ij} - \frac{1}{2C^2}g_{ij}), \tag{2.9}
$$

the tensor $H_{ij\ldots k}$ is the volume form of S_{D-2}, i.e.

$$
H_{ij\cdots k} = -\sqrt{h}\epsilon_{ij\cdots k}. \tag{2.10}
$$

Here the scalars are given by

$$
\begin{aligned}
\eta_0 &= \frac{C^{2(D-2)}}{(D-3)!}\left(\frac{A''}{AB^2} - \frac{A'B'}{AB^3} - \frac{C''}{CB^2} + \frac{B'C'}{B^3C}\right), \\
\eta_1 &= \frac{A'C'}{AC} - \frac{C''}{C} + \frac{B'C'}{BC}, \\
\eta_2 &= \frac{C^{2(D-3)}}{(D-4)!}\left(1 - \frac{C'^2}{B^2} + \frac{A'C'C}{AB^2} - \frac{A''C^2}{AB^2} + \frac{A'B'C^2}{AB^3} + \frac{CC''}{B^2} - \frac{B'C'C}{B^3}\right),
\end{aligned} \tag{2.11}
$$

$$\eta_3 = -\frac{A'C'}{AB^2C}. \tag{2.12}$$

The Ricci tensor, Ricci scalar and Einstein tensor can be computed from Riemann as follows:

$$\begin{aligned} R_{ij} &= \left[\frac{(D-3)!}{2C^{2(D-2)}}(\eta_0(D-4)+\eta_2)+\frac{1}{B^2}\eta_1+\eta_3(D-1)\right]g_{ij} \\ &\quad +\eta_1(D-2)k_ik_j+[\eta_0(D-2)+\eta_2]M_{ij}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} R &= \frac{(D-3)!}{C^{2(D-2)}}[\eta_0(D-4)(D-1)+\eta_2(D-2)]+\frac{2\eta_1(D-1)}{B^2} \\ &\quad +\eta_3 D(D-1), \end{aligned} \tag{2.14}$$

$$\begin{aligned} G_{ij} &= -\left[\frac{(D-3)!}{2C^{2(D-2)}}(\eta_0(D-4)(D-2)+\eta_2(D-3))+\frac{1}{B^2}\eta_1(D-2)\right. \\ &\quad \left.+\frac{1}{2}\eta_3(D-1)(D-2)\right]g_{ij}+\eta_1(D-2)k_ik_j \\ &\quad +[\eta_0(D-2)+\eta_2]M_{ij}. \end{aligned} \tag{2.15}$$

The covariant derivatives of $H_{ij\ldots k}$ and k_i are given as

$$\nabla_l H_{ij\ldots m} = -\rho[(D-2)H_{ij\ldots m}k_l + k_iH_{lj\ldots m} + k_jH_{il\ldots m} + \ldots + k_mH_{ij\ldots l}], \tag{2.16}$$

$$\nabla_i k_j = \rho_1 g_{ij} + \rho_2 M_{ij} + \rho_3 k_ik_j \tag{2.17}$$

where

$$\begin{aligned} \rho &= \frac{C'}{C} & \rho_1 &= \frac{A'C+AC'}{2AB^2C} \\ \rho_2 &= \frac{C^{2(D-2)}}{(D-3)!}\frac{AC'-A'C}{AB^2C} & \rho_3 &= -\frac{(AB)'}{AB}. \end{aligned} \tag{2.18}$$

The covariant derivatives of $H_{ij\cdots k}$ and k_i are expressed only in terms of themselves and the metric tensor. Riemann tensor is given in terms of $H_{ij\cdots k}$, g_{ij} and k_i. Hence we have the following theorem:

Theorem 1 *Covariant derivatives of the Riemann tensor R_{ijkl}, the tensor H_{ij} and the vector k_i at any order are expressible only in terms of $H_{ij\cdots k}$, g_{ij} , k_i.*

Since contraction of k^i with $H_{ij\cdots k}$ vanishes, the only symmetric tensors constructable out of $H_{ij\cdots k}$, g_{ij} and k_i are M_{ij}, the metric tensor g_{ij} and $k_i k_j$. Then the following theorems hold:

Theorem 2 *Any second rank symmetric tensor constructed out of the Riemann tensor, anti-symmetric tensor $H_{ij\cdots k}$, dilaton field $\phi = \phi(r)$ and their covariant derivatives is a linear combination of M_{ij} , g_{ij} and $k_i k_j$.*

Let this symmetric tensor be E'_{ij}. Then we have

$$E'_{ij} = \sigma_1 M_{ij} + \sigma_2 g_{ij} + \sigma_3 k_i k_j \tag{2.19}$$

where σ_1, σ_2 and σ_3 are scalars which are functions of the metric functions, invariants constructed out of the curvature tensor R_{ijkl}, H_{ij} and the dilaton field.

Theorem 3 *Any vector constructed out of the Riemannian tensor R_{ijkl}, $H_{ij\ldots k}$ the dilaton field $\phi = \phi(r)$ and their covariant derivatives is proportional to k_i.*

Let this vector be E'_i. Hence

$$E'_i = \sigma k_i \tag{2.20}$$

where σ is a scalar like σ_1, σ_2, σ_3.

Theorem 2 has an application:

Theorem 4 *In a spherically symmetric, D-dimensional spacetime, the coefficients a_i in the identity*

$$a_1 R_{imnl} R_j^{mnl} + a_2 R_{im} R_j^m + a_3 R_{jmin} R^{mn} + a_4 R_{ij} + a_5 g_{ij} = 0 \tag{2.21}$$

can be found in terms of the η_i.

This can be easily seen if we write each of the symmetric tensors as a combination of M_{ij}, g_{ij}, $k_i k_j$. Let us denote these tensors as follows:

$$
\begin{aligned}
R_i^a R_{ja} &= \beta_1 g_{ij} + \beta_2 M_{ij} + \beta_3 k_i k_j \\
R_{jaib} R^{ab} &= \beta_4 g_{ij} + \beta_5 M_{ij} + \beta_6 k_i k_j \\
R_i^{abc} R_{jabc} &= \beta_7 g_{ij} + \beta_8 M_{ij} + \beta_9 k_i k_j \\
R_{ij} &= f_1 g_{ij} + f_2 M_{ij} + f_3 k_i k_j .
\end{aligned}
\tag{2.22}
$$

Thus, we obtain

$$
\begin{aligned}
\beta_7 a_1 + \beta_1 a_2 + \beta_4 a_3 + f_1 a_4 + a_5 &= 0, \\
\beta_8 a_1 + \beta_2 a_2 + \beta_5 a_3 + f_2 a_4 &= 0, \\
\beta_9 a_1 + \beta_3 a_2 + \beta_6 a_3 + f_3 a_4 &= 0.
\end{aligned}
\tag{2.23}
$$

Provided we know a_4 and a_5, we can compute a_1, a_2, a_3.

Let's denote

$$
\begin{vmatrix} \beta_7 & \beta_1 & \beta_4 \\ \beta_8 & \beta_2 & \beta_5 \\ \beta_9 & \beta_3 & \beta_6 \end{vmatrix} = \frac{1}{\Psi},
\tag{2.24}
$$

then,

$$
a_1 = -\Psi[(\beta_2\beta_6 - \beta_3\beta_5)(f_1 a_4 + a_5) + (\beta_3\beta_4 - \beta_1\beta_6)(f_2 a_4) + (\beta_1\beta_5 - \beta_2\beta_4)(f_3 a_4)],
\tag{2.25}
$$

$$
a_2 = -\Psi[(\beta_5\beta_9 - \beta_6\beta_8)(f_1 a_4 + a_5) + (\beta_6\beta_7 - \beta_4\beta_8)(f_2 a_4) + (\beta_4\beta_8 - \beta_5\beta_7)(f_3 a_4)],
\tag{2.26}
$$

$$a_3 = -\Psi[(\beta_3\beta_8 - \beta_2\beta_9)(f_1 a_4 + a_5) + (\beta_1\beta_9 - \beta_3\beta_7)(f_2 a_4) + (\beta_2\beta_7 - \beta_1\beta_8)(f_3 a_4)]. \tag{2.27}$$

We will give β's explicitly in appendix.

We may also write the Riemann tensor in the form

$$R_{ijml} = g_{jl}S_{im} - g_{jm}S_{il} + g_{im}S_{jl} - g_{il}S_{mj} + e_2 F_{ij} F_{ml}, \tag{2.28}$$

where

$$F_{ij} = \frac{AB}{C^2}(t_i k_j - t_j k_i). \tag{2.29}$$

Other tensors are defined similar to the previous case, i.e.

$$S_{ij} = e_0 \widetilde{M_{ij}} + e_1 k_i k_j + \frac{1}{2} e_3 g_{ij}, \tag{2.30}$$

$$\widetilde{M_{ij}} = F_{mj} F_i^m - \frac{1}{4} F^2 g_{ij} = \frac{1}{C^2}(h_{ij} - \frac{1}{2C^2} g_{ij}). \tag{2.31}$$

The scalars are given as

$$\begin{aligned} e_0 &= C^2 - \frac{C'^2 C^2}{B^2} + \frac{C^3 C' A'}{A B^2}, \\ e_1 &= \eta_1, \\ e_2 &= -C^2\left(1 - \frac{C'^2}{B^2} + \frac{A'C'C}{AB^2} - \frac{A''C^2}{AB^2} + \frac{A'B'C^2}{AB^3} + \frac{CC''}{B^2} - \frac{B'C'C}{B^3}\right), \\ e_3 &= \eta_3. \end{aligned} \tag{2.32}$$

Notice that when $D = 4$, $\widetilde{M_{ij}} = M_{ij}$, and $e_2 = -\eta_2$.

The covariant derivatives of F_{ij} , k_i and t_i are given as

$$\nabla_l F_{ij} = -\frac{3C'}{C} F_{ij} k_l - \frac{AC'}{BC^3}(t_j g_{il} - t_i g_{jl}), \tag{2.33}$$

$$\nabla_i k_j = \rho_1 g_{ij} + \tilde{\rho}_2 \widetilde{M_{ij}} + \rho_3 k_i k_j, \tag{2.34}$$

$$\nabla_i t_j = \frac{A'}{A}(t_i k_j + t_j k_i). \tag{2.35}$$

where $\tilde{\rho}_2 = \frac{(D-3)!}{C^{2(D-4)}}\rho_2$. The covariant derivatives of F_{ij} and k_i are expressed in terms of themselves, metric tensor and t_i. So we have a similar theorem for an extended set:

Theorem 5 *Covariant derivatives of the Riemann tensor R_{ijkl}, the anti-symmetric tensor F_{ij}, and the vectors k_i and t_i at any order are expressible only in terms of F_{ij}, k_i, t_i and g_{ij}.*

Chapter 3

Schwarzschild and Reissner-Nordström Solutions

These are the earliest spherically symmetric solutions, obtained in 1916, so they deserve a short rewiew here.

Let's take the line element as

$$ds^2 = -A^2dt^2 + B^2dr^2 + r^2d\Omega^2. \tag{3.1}$$

In other words, take $C(r) = r$. Our equation is

$$G_{ij} = 0. \tag{3.2}$$

We have already computed G_{ij} in terms of g_{ij}, k_ik_j and M_{ij} (2.15), so equating the coefficients of these terms to zero, we obtain three equations:

$$\frac{(D-3)!}{2C^{2(D-2)}}[\eta_0(D-4)(D-2)+\eta_2(D-3)]+\frac{\eta_1}{B^2}(D-2)+\frac{\eta_3}{2}(D-1)(D-2)=0, \tag{3.3}$$

$$\eta_1(D-2) = 0, \tag{3.4}$$

$$\eta_0(D-2) + \eta_2 = 0, \tag{3.5}$$

(3.4) gives

$$\frac{A'}{A} + \frac{B'}{B} = 0 \tag{3.6}$$

so $AB = c_1$ but the boundary condition for the metric to be Lorentzian at infinity requires that $c_1 = 1$.

Thus

$$AB = 1. \tag{3.7}$$

Using (3.5), we can find η_2 in terms of η_0. Inserting η_2 and substituting $B = \frac{1}{A}$ in (3.3) gives

$$AA'' + A'^2 + \frac{(D-2)}{r} AA' = 0, \tag{3.8}$$

which is actually

$$u' + (D-2)\frac{u}{r} = 0 \tag{3.9}$$

where $u = A\,A'$.

Thus $A^2 = c_2 + c_3 r^{-D+3}$. Inserting this in (3.5), we find that $c_2 = 1$,

so

$$\begin{aligned} A^2 &= 1 - \frac{2m}{r^{D-3}}, \\ B^2 &= \left(1 - \frac{2m}{r^{D-3}}\right)^{-1}, \end{aligned} \tag{3.10}$$

where $2m$ is the integration constant c_3.

So we obtain the Schwarzschild solution. For Reissner-Nordström solution we have to take electrostatic charge Q into account.

Our equation is

$$G_{ij} = T_{ij} \tag{3.11}$$

where

$$T_{ij} = -2\left[F_{iN}F_j^N - \frac{1}{4}F_{MN}F^{MN}g_{ij}\right], \tag{3.12}$$

$$F_{tr} = Q\,\frac{AB}{r^{D-2}}. \tag{3.13}$$

T_{ij} turns out to be

$$T_{ij} = -2\,Q^2\frac{r^{2\,(D-4)}}{(D-3)!}\,M_{ij}. \tag{3.14}$$

Thus we obtain the same set of equations except for(3.5).

Proceeding in a similar manner, instead of (3.8) we obtain

$$AA'' + A'^2 + \frac{(D-2)}{r}AA' = -2\frac{(D-3)}{(D-2)}\frac{Q^2}{r^{2(D-2)}}, \tag{3.15}$$

which is actually

$$\frac{(r^{D-2}\,AA')'}{r^{D-2}} = -2\frac{(D-3)}{(D-2)}\frac{Q^2}{r^{2(D-2)}}, \tag{3.16}$$

so we find

$$A^2 = \frac{-2Q^2}{(D-2)(D-3)}\,r^{2(D-3)}. \tag{3.17}$$

But this is the non-homogeneous solution, we have to add the solution of (3.8) to this.

Thus

$$
\begin{aligned}
A^2 &= 1 - \frac{2m}{r^{D-3}} - \frac{2\,Q^2}{(D-2)\,(D-3)\,r^{2(D-3)}}, \\
B^2 &= \left(1 - \frac{2m}{r^{D-3}} - \frac{2\,Q^2}{(D-2)(D-3)\,r^{2(D-3)}}\right)^{-1}
\end{aligned}
\tag{3.18}
$$

Here integration constants are found by the asymptotic behaviour of the solution:

$$
\begin{aligned}
\lim_{r\to\infty} r^{D-3}(A^2-1) &= -2M, \\
\lim_{r\to\infty} r^{D-2} F_{tr} &= Q.
\end{aligned}
\tag{3.19}
$$

Chapter 4

Levi-Civita Bertotti-Robinson Metric

4.1 Einstein's equation

An example is the Levi-Civita Bertotti-Robinson (LCBR) metric in D-dimensions

$$g_{ij} = \frac{q^2}{r^2}\left(-t_i t_j + c_0^2 k_i k_j + r^2 h_{ij}\right) \tag{4.1}$$

where h_{ij} is the metric on S_{D-2}, $t_i = \delta_i^t$, $k_i = \delta_i^r$, q and c_0 are constants.

This is the previous metric with the choice $A = \frac{q}{r}$, $B = \frac{qc_0}{r}$, $C = q$, then

$$F_{ij} = \frac{c_0}{r^2}(t_i k_j - t_j k_i), \tag{4.2}$$

$$\widetilde{M_{ij}} = F_{mj} F_i^m - \frac{1}{4} F^2 g_{ij}, \tag{4.3}$$

$e_1 = e_3 = 0 \Rightarrow S_{ij} = e_0 \widetilde{M_{ij}}$, $e_0 = q^2$,

$$R_{ijkl} = q^2[g_{jl}\widetilde{M_{ik}} - g_{jk}\widetilde{M_{il}} + g_{ik}\widetilde{M_{jl}} - g_{il}\widetilde{M_{kj}}] + \frac{q^2(1-c_0^2)}{c_0^2} F_{ij} F_{ml}, \tag{4.4}$$

$$R_{ij} = q^2[(D-3) + \frac{1}{c_0^2}]\widetilde{M_{ij}} + \frac{1}{2q^2}[(D-3) - \frac{1}{c_0^2}]g_{ij}, \tag{4.5}$$

$$G_{ij} = q^2[(D-3) + \frac{1}{c_0^2}]\widetilde{M_{ij}} + \frac{1}{2q^2}[\frac{1}{c_0^2} - (D-3)^2]g_{ij}. \tag{4.6}$$

It is easily seen that

$$\nabla_l F_{ij} = 0 \quad \nabla_m R_{ijkl} = 0. \tag{4.7}$$

If we consider the Einstein's equations

$$G_{ij} = T_{ij} \tag{4.8}$$

where

$$T_{ij} = \widetilde{F_{mj}}\widetilde{F_i^m} - \frac{1}{4}\widetilde{F^2}\, g_{ij} \tag{4.9}$$

and $\widetilde{F_{ij}}(E.M) = eF_{ij}$, we obtain

$$\begin{aligned} q^2\left[(D-3) + \frac{1}{c_0^2}\right] &= e^2, \\ \frac{1}{2q^2}\left[\frac{1}{c_0^2} - (D-3)^2\right] &= \lambda \end{aligned} \tag{4.10}$$

where λ is the cosmological constant. To eliminate the cosmological constant, we let $\frac{1}{c_0} = |D-3|$. It is interesting that spacetime is conformally flat only for $D = 4$.

We can obtain the same equations using the expression (2.15). This time we have to choose

$$T_{ij} = \widetilde{H}_{im\cdots n}\widetilde{H}_j^{m\cdots n} - \frac{1}{2(D-2)}\widetilde{H^2}g_{ij} \tag{4.11}$$

and $\widetilde{H}_{ij\cdots k} = eH_{ij\cdots k}$. When the cosmological constant is set equal to zero, we obtain

$$c_0^2 = \frac{1}{(D-3)^2}. \tag{4.12}$$

Hence the higher dimensional ($D > 4$) Levi-Civita Bertotti-Robinson space-times without cosmological constants can not be conformally flat.

4.2 The Solutions of Lovelock Theory

In the following $\delta^{i_1 i_2 \ldots i_N}_{j_1 j_2 \ldots j_N}$ is the generalized Kronecker delta defined by

$$\delta^{i_1 i_2 \ldots i_N}_{j_1 j_2 \ldots j_N} = det \begin{vmatrix} \delta^{i_1}_{j_1} & & \delta^{i_1}_{j_N} \\ \vdots & & \vdots \\ \delta^{i_N}_{j_1} & \ldots & \delta^{i_N}_{j_N} \end{vmatrix} \tag{4.13}$$

According to a theorem by Lovelock, the only symmetric tensor $A^{ij} = A^{ij}(g_{rs}; g_{rs,t} : g_{rs,tu})$ for which $A^{ij}_{,j} = 0$ is

$$A^i_j = \sum_{p=1}^{m-1} a_p \delta^{ih_1 \ldots h_{2p}}_{jj_1 \ldots j_{2p}} R^{j_1 j_2}_{h_1 h_2} R^{j_3 j_4}_{h_3 h_4} \cdots R^{j_{2p-1} j_{2p}}_{h_{2p-1} h_{2p}} + a\delta^i_j \tag{4.14}$$

where a and a_p are arbitrary constants. We have found the equation for $n = 5, 6$

$$a_1 \delta^{in_1 n_2}_{jm_1 m_2} R^{m_1 m_2}_{n_1 n_2} + a_2 \delta^{in_1 n_2 n_3 n_4}_{jm_1 m_2 m_3 m_4} R^{m_1 m_2}_{n_1 n_2} R^{m_3 m_4}_{n_3 n_4} + a\delta^i_j = 0 \tag{4.15}$$

which is

$$G_{ij} + \alpha_0[(R^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd})g_{ij} + 4(2R^a_i R_{ja} + 2R_{jaib}R^{ab} - RR_{ij} - R^{abc}_i R_{jabc})] = \lambda g_{ij} \tag{4.16}$$

where $\alpha_0 = -\frac{a_2}{a_1}$

We know that every symmetric tensor of rank two can be written as a linear combination of g_{ij}, M_{ij} and $k_i k_j$. To calculate these we need the following:

For the LCBR metric

$$ds^2 = \frac{q^2}{r^2}\left(-dt^2 + c_0^2\,dr^2 + r^2 d\Omega^2\right) \tag{4.17}$$

where q and c_0 are constants, we have

$$\begin{aligned} \eta_0 &= \frac{1}{\alpha c_0^2}, \\ \eta_1 &= 0, \\ \eta_2 &= \frac{(D-3)(c_0^2-1)}{\alpha c_0^2}, \\ \eta_3 &= 0. \end{aligned} \tag{4.18}$$

If we insert these values in t_1, t_2, t_3 we find

$$t_1 = \frac{1}{2q^2c_0^2} - \frac{(D-3)^2}{2q^2} + \frac{\alpha_0}{q^4}(D-3)(D-4)^2(D-5) - \frac{2\alpha_0}{q^4c_0^2}(D-3)(D-4), \tag{4.19}$$

$$t_2 = \frac{1}{\alpha}\left(D-3+\frac{1}{c_0^2}\right) - \frac{4\alpha_0(D-3)(D-4)}{\alpha q^2 c_0^2}\left[1 + c_0^2(D-5)\right], \tag{4.20}$$

$$t_3 = 0. \tag{4.21}$$

Field equations reduce to $t_1 = \lambda$, $t_2 = t_3 = 0$. This gives relations between constants of the theory and the constants of the metric.

For $D = 5$ we obtain

$$\begin{aligned} q^2 - 4q^2c_0^2 - 8\alpha_0 &= 2q^4c_0^2\,\lambda, \\ q^2 + 2q^2c_0^2 - 8\alpha_0 &= 0 \end{aligned} \tag{4.22}$$

which gives $c_0^2 = 0$ or $q^2 = -\frac{3}{\lambda}$

For $D = 6$ we obtain

$$
\begin{aligned}
q^2(1 - 9c_0^2) + 24\alpha_0(c_0^2 - 1) &= 2q^4 c_0^2 \lambda, \\
3q^2 c_0^2 + q^2 - 24\alpha_0(1 + c_0^2) &= 0.
\end{aligned}
\tag{4.23}
$$

which gives $q^2 = 4\alpha_0$, $c_0^2 = -\frac{5}{3}$ for $\lambda = 0$ and

$$
\begin{aligned}
q^2 &= \frac{-3 + \sqrt{9 + 24\alpha_0\lambda}}{\lambda}, \\
c_0^2 &= \frac{q^2 - 24\alpha_0}{q^2(3 + \lambda)},
\end{aligned}
\tag{4.24}
$$

for $\lambda \neq 0$.

So there is no solution for 5 or 6 dimensional spacetimes without a cosmological constant.

4.3 Solution of the most general theory

The Lagrangian of the most general theory will be a scalar containing the Riemann tensor, metric tensor, and their derivatives, contractions and multiple products of all orders. But, according to theorem 2, all second rank symmetric tensors constructed out of these will be expressible in terms of *g_{ij}, M_{ij}, and $k_i k_j$.*

So, whatever the theory is, we'll obtain two equations for the LCBR metric, because the coefficient of the $k_i k_j$ term will automatically vanish. This will give us two algebraic equations for two unknown constants in the metric, namely q and c_0. These equations may or may not have a solution according to theory. For example, in the preceding case, in $D = 5, 6$ we have no solution.

Chapter 5

Solutions of the Low Energy Limit of the String Theory

The gravitational field equations obtained from the low energy limit of the string theory can be obtained from the following lagrangian

$$L = \sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{4}{(D-2)\kappa^2}(\nabla\phi)^2 - \frac{1}{4}\, e^{-\alpha_e\phi}F^2\right]. \qquad (5.1)$$

The field equations are

$$G_{ij} = \frac{8}{(D-2)}\left[\partial_i\phi\partial_j\phi - \frac{1}{2}(\nabla\phi)^2 g_{ij}\right] - \kappa^2 e^{-\alpha_e\phi}\left[F_i^m F_{jm} - \frac{1}{4}F^2 g_{ij}\right], \qquad (5.2)$$

$$\nabla_i(e^{-\alpha_e\phi}F^{ij}) = 0, \qquad (5.3)$$

$$\partial_i(\sqrt{-g}\, g^{ij}\, \partial_j\phi) + \frac{(D-2)\kappa^2\alpha_e\sqrt{-g}}{32} e^{-\alpha_e\phi}F^2 = 0, \qquad (5.4)$$

where F_{ij} is the Maxwell and ϕ is the dilaton field. Here $i, j = 1, 2, ..., D \geq 4$.

In static spherically symmetric spacetimes, gravitational field equations first lead to

$$\frac{\eta_0(D-4)(D-2)!}{2C^{2(D-2)}}+\frac{\eta_2(D-3)(D-3)!}{2C^{2(D-2)}}+\frac{\eta_1(D-2)}{B^2}+\frac{\eta_3(D-1)(D-2)}{2}$$
$$-\frac{4\phi'^2}{(D-2)B^2}=0, \tag{5.5}$$

$$\eta_1(D-2)-\frac{8\phi'^2}{(D-2)}=0, \tag{5.6}$$

$$\eta_0(D-2)!+\eta_2(D-3)!-\frac{\kappa^2 Q^2 e^{\alpha_e\phi}}{A_{D-2}^2}=0. \tag{5.7}$$

Dilaton equation is:

$$\frac{8}{D-2}\left[\frac{A}{B}\,C^{D-2}\,\phi'\right]'-\frac{\alpha_e\,A\,B\,\kappa^2\,Q^2\,e^{\alpha_e\phi}}{2\,C^{D-2}\,A_{D-2}^2}=0. \tag{5.8}$$

From (5.5) and (5.6) we obtain

$$\left[\frac{\left(A\,C^d\right)'\,C}{B}\right]'=d^2\,A\,B\,C^{d-1} \tag{5.9}$$

where $d=D-3$. Using the freedom in choosing the r coordinate we can let

$$A\,B\,C^{d-1}=r^{d-1}, \tag{5.10}$$

by using (5.9) and (5.10) we obtain

$$A^2\,C^{2d}=r^{2d}-2b_1\,r^d+b_2 \tag{5.11}$$

where b_1 and b_2 are integration constants.

A combination of the dilaton (5.3) and gravitational field equations (5.2) gives

$$d\,T^2 - \frac{d-1}{r}\,T + T' + \frac{8}{(d+1)^2}\,\phi'^2 = 0, \tag{5.12}$$

where T is defined as

$$T = \frac{(r^d + c_1)\,r^{d-1}}{(r^{2d} - 2b_1 r^d + b_2)} - \frac{16\phi'}{(d+1)^2\alpha_e}. \tag{5.13}$$

Defining now

$$\alpha_e\,\phi' = \frac{k_1\,d\,r^{d-1}}{(r^{2d} - 2b_1 r^d + b_2)}\,\psi(\rho), \tag{5.14}$$

the equation (5.12) becomes

$$\frac{r^{2d} - 2b_1 r^d + b_2}{d\,r^{d-1}}\,\frac{d\psi}{d\rho}\,\frac{d\rho}{dr} = (\psi + \mu)^2 + \nu^2. \tag{5.15}$$

The constants are given by

$$\begin{aligned} a &= \frac{(d+1)^2\,\alpha_e^2}{32\,d}, \\ \mu &= -(c_1 + b_1), \\ \nu^2 &= a\mu^2 - \Delta(a+1). \end{aligned} \tag{5.16}$$

Now, if we solve the auxiliary equation

$$\frac{r^{2d} - 2b_1 r^d + b_2}{d\,r^{d-1}}\frac{d\rho}{dr} = \rho \tag{5.17}$$

and insert ρ above, we obtain

$$\rho\frac{d\psi}{d\rho} = (\psi + \mu)^2 + \nu^2. \tag{5.18}$$

Note that ρ depends on the sign of $\Delta = b_1^2 - b_2$. ϕ can be found from ψ by

$$\phi = \frac{k_1}{\alpha_e} \int \frac{\psi(\rho)}{\rho} d\rho + \phi_0, \tag{5.19}$$

where $k_1 = \frac{2a}{a+1}$. The metric function C is connected to ϕ as

$$\frac{C'}{C} = \frac{(r^d + c_1) r^{d-1}}{(r^{2d} - 2b_1 r^d + b_2)} - \frac{16\phi'}{(d+1)^2 \alpha_e} \tag{5.20}$$

this gives us

$$\ln(\frac{C}{c_0})^d = \left(\int \frac{u + b_1 + c_1}{u^2 - \Delta} du \right) - \frac{\alpha_e}{2a} \phi \tag{5.21}$$

where $u = r^d - b_1$.

Metric functions A and B can be found from C through the equations (5.10) and (5.11).

We have three different cases according to the sign of Δ,

$$\begin{array}{lll} \text{Case 1} & \Delta > 0 & \ln \rho(r) = \frac{1}{r_1^d - r_2^d} \ln \left(\frac{r^d - r_1^d}{r^d - r_2^d} \right), \\ \text{Case 2} & \Delta = 0 & \ln \rho(r) = -\frac{1}{r^d - r_3^d}, \\ \text{Case 3} & \Delta < 0 & \ln \rho(r) = \frac{1}{\sqrt{-\Delta}} \left[\arctan \left(\frac{r^d - b_1}{\sqrt{-\Delta}} \right) - \frac{\pi}{2} \right] \end{array} \tag{5.22}$$

where $r_1^d = b_1 + \sqrt{\Delta}$, $r_2^d = b_1 - \sqrt{\Delta}$, $r_3^d = b_1$.

The integration constants c_0 and ϕ_0 are determined through the asymptotic behaviour of the functions $C(r)$ and $\phi(r)$.

$$\lim_{r \to \infty} \phi = \phi_0 = 0, \tag{5.23}$$

$$\lim_{r \to \infty} \frac{C(r)}{r} = c_0 = 1. \tag{5.24}$$

To determine the remaining integration constants, we use the asymptotic behaviour of the metric, scalar field and the tensor field F_{ij} as follows:

$$\begin{aligned}
\lim_{r\to\infty} r^d (A^2 - 1) &= -\frac{2\kappa^2 M}{A_{d+1}(d+1)}, \\
\lim_{r\to\infty} r^{d+1}\phi' &= -\frac{\kappa\sqrt{d+1}}{2A_{d+1}}\Sigma, \\
\lim_{r\to\infty} r^{d+1} F_{tr} &= \frac{Q}{A_{d+1}}.
\end{aligned} \tag{5.25}$$

CASE 1

$$\Delta > 0 \tag{5.26}$$

Which means there are two roots to the equation (5.11). According to the sign of ν^2 we have three distinct solutions

Type 1

$$\begin{aligned}
\nu^2 &< 0 \qquad\qquad \lambda^2 = -\nu^2, \\
\psi &= \frac{\lambda - \mu + c_2(\lambda+\mu)\rho^{2\lambda}}{1 - c_2\rho^{2\lambda}}.
\end{aligned} \tag{5.27}$$

The metric functions become

$$\begin{aligned}
A^2 &= \frac{(r^d - r_1^d)(r^d - r_2^d)}{C^{2d}}, \qquad B^2 = \frac{r^{2d-2}\,C^2}{(r^d - r_1^d)(r^d - r_2^d)}, \\
C^d &= (r^d - r_2^d)\left(\frac{1 - c_2\,\rho^{2\lambda}}{1 - c_2}\right)^{k_2} \rho^{k_3}.
\end{aligned} \tag{5.28}$$

Dilaton field is given as

$$e^{\alpha_e \phi} = \left[\frac{(1-c_2)\,\rho^{\lambda-\mu}}{1 - c_2\,\rho^{2\lambda}}\right]^{k_1} \tag{5.29}$$

The constants are given by

$$k_2 = \frac{1}{(a+1)}, \qquad k_3 = \frac{(r_1^d - r_2^d)}{2} - \frac{(\nu + a\mu)}{(a+1)}. \tag{5.30}$$

Undetermined integration constants are r_1, r_2, c_1 and c_2. From the boundary condition (5.25) we find that

$$\begin{aligned} 2\,M &= \left[\frac{1+c_2}{1-c_2}\,\lambda + a\,\mu\right] e_1, \\ \Sigma &= \left[\frac{1+c_2}{1-c_2}\,\lambda - \,\mu\right] e_2, \\ Q &= \lambda\,\frac{\sqrt{c_2}}{1-c_2} e_3. \end{aligned} \tag{5.31}$$

The constants e_i are given by

$$\begin{aligned} e_1 &= \frac{2\,A_{d+1}\,(d+1)}{(a+1)\kappa^2}, \\ e_2 &= \frac{A_{d+1}\,\alpha_e\,(d+1)^{\frac{3}{2}}}{8\,(a+1)\kappa}, \\ e_3 &= \frac{2\,A_{d+1}}{\kappa}\sqrt{\frac{(d+1)\,d}{(a+1)}}. \end{aligned} \tag{5.32}$$

Note that, we have four integration constants (c_1, c_2, r_1^d and r_2^d) but there exists only three equations to determine them. Also note that c_1 does not appear in the solution directly, so we have a freedom in c_1. ($c_2 \neq 1$) In order to complete the solution,we need to determine the integration constants in terms of the physical parameters M, Σ and Q. Let us define some auxiliary variables to solve the set of algebraic equations (5.31)

$$\begin{aligned} T_1 &= \frac{1}{a+1}(\frac{a\,\Sigma}{e_2} + \frac{2\,M}{e_1}), \qquad T_2 = \frac{1}{a+1}(\frac{2\,M}{e_1} - \frac{\Sigma}{e_2}), \\ T_3 &= \sqrt{1 - \frac{4\,Q^2}{e_3^2\,T_1^2}}. \end{aligned} \tag{5.33}$$

Then the integration constants are

$$
\begin{aligned}
\lambda &= \pm T_3 T_1, \\
\mu &= T_2, \\
c_2 &= -1 + \frac{2}{1 \pm T_3}.
\end{aligned}
\tag{5.34}
$$

and

$$
\Delta = \frac{a\mu^2 + \lambda^2}{a+1} \tag{5.35}
$$

The reality of T_3 imposes

$$
M + g\,\Sigma \geq s\,|Q| \tag{5.36}
$$

where g and s are given by

$$
g = \frac{\alpha_e\,(d+1)^{3/2}}{4d\,\kappa} \quad , \quad s = \frac{1}{\kappa}\sqrt{\frac{(d+1)(a+1)}{d}} \tag{5.37}
$$

Such an inequality has been found by Gibbons and Wells for D=4.

When $\Delta > 0$ we have two roots. In general these roots are the singular points of the space-time. If the integration constants satisfy some additional constraints one of these roots becomes regular. In this case we have a black hole solution carrying mass M, electric charge Q and scalar charge Σ. An invariant of the space-time is the scalar curvature is given by

$$
R = A_1 \frac{\rho^{z_1}\left[-\mu + \lambda + \frac{2c_2\lambda\rho^{2\lambda}}{1-c_2\rho^{2\lambda}}\right]^2}{[(r^d - r_1^d)(r^d - r_2^d)]^{1+\frac{1}{d}}(1 - c_2\rho^{2\lambda})^{\frac{2k_2}{d}}} + A_2 \frac{\rho^{z_1+2\lambda}\left(\frac{1-c_2}{1-c_2\rho^{2\lambda}}\right)^{\frac{2k_2}{d}+2}}{[(r^d - r_1^d)(r^d - r_2^d)]^{1+\frac{1}{d}}} \tag{5.38}
$$

where

$$\begin{aligned}
z_1 &= \frac{2}{d(a+1)}(a\mu+\lambda), \\
A_1 &= \frac{8}{d+1}\frac{k_1^2 d^2}{\alpha_e^2}, \\
A_2 &= \frac{\kappa^2}{2}\frac{(D-4)}{(D-2)}.
\end{aligned} \tag{5.39}$$

As $r \to r_1$ we have a singularity unless we choose $\mu = \lambda$. By this choice, $\mu = \frac{r_1^d - r_2^d}{2}$ and hence $\rho^{z_1} \sim (r^d - r_1^d)^{\frac{2}{d}}$, $\rho^{2\lambda} \sim (r^d - r_1^d)^2$ around the horizon, so $R \to 0$.

If we insert these values for μ and λ in the solution, we obtain

$$C^d = (r^d - r_2^d)\left(1 - \frac{c_2}{1-c_2}\frac{(r_2^d - r_1^d)}{r^d - r_2^d}\right)^{k_2} \tag{5.40}$$

At this point, the choice of $r_2 = 0$ gives Gibbons-Maeda solution, whereas the choice of $c_2 = \frac{r_2^d}{r_1^d}$ gives the GHS solution. It is easy to show that Gibbons-Maeda metric is the same as the GHS metric with $r > r_2$.

Type 2

$$\begin{aligned}
\nu^2 &> 0 \\
\psi &= \nu\,\tan(c_2 + \nu \ln\rho) - \mu
\end{aligned} \tag{5.41}$$

$$\int \frac{\psi(\rho)}{\rho} d\rho = -\ln\left[\cos(c_2 + \nu\ln\rho)\right] - \mu\ln\rho + c_3 \tag{5.42}$$

After similar steps as the previous type, we arrive at the solution

$$A^2 = \frac{(r^d - r_1^d)(r^d - r_2^d)}{C^{2d}}, \qquad B^2 = \frac{r^{2d-2}\,C^2}{(r^d - r_1^d)(r^d - r_2^d)}, \tag{5.43}$$

$$C^d = (r^d - r_2^d)\,\rho^{\frac{r_1^d - r_2^d}{2} - \frac{\mu k_1}{2}}\left[\frac{\cos(c_2 + \nu\ln\rho)}{\cos c_2}\right]^{k_2} \tag{5.44}$$

Scalar field is given as

$$e^{\alpha_e \phi} = \left[\frac{\cos c_2}{\rho^{\mu}\,\cos(c_2 + \nu \ln \rho)}\right]^{k_1} \tag{5.45}$$

Physical parameters are found using (5.25)

$$\begin{aligned} 2M &= (a\mu + \nu \tan c_2)\, e_1, \\ \Sigma &= (-\mu + \nu \tan c_2)\, e_2, \\ Q &= \frac{\nu}{\cos c_2}\, e_3, \end{aligned} \tag{5.46}$$

$$\begin{aligned} \mu &= T_2, \\ \sin c_2 &= \frac{e_3\, T_1}{Q}, \\ \nu \tan c_2 &= T_1. \end{aligned} \tag{5.47}$$

The condition $|\sin c_2| < 1$ imposes

$$M + g\,\Sigma < s\,|Q| \tag{5.48}$$

where g and s are defined in (5.37). We also have to check the sign of Δ.

$$(a+1)\Delta = \frac{1}{a+1}\left(\frac{a\,\Sigma^2}{e_2^2} + \frac{4\,M^2}{e_1^2}\right) - \frac{Q^2}{e_3^2}. \tag{5.49}$$

Here the sign of Δ puts a constraint on the physical variables.

Type 3

$$\begin{aligned} \nu^2 &= 0, \\ \psi &= -\mu - \frac{1}{\ln \rho + c_2}, \end{aligned} \tag{5.50}$$

$$\int \frac{\psi(\rho)}{\rho} d\rho = -\ln(\ln \rho + c_2) - \mu \ln \rho + c_3, \tag{5.51}$$

$$e^{\alpha_e \phi} = \left[\frac{c_2}{\rho^{\mu}(c_2 + \ln \rho)} \right]^{k_1}, \tag{5.52}$$

$$C^d = \left(\frac{c_2 + \ln \rho}{c_2} \right)^{k_2} [(r^d - r_1^d)(r^d - r_2^d)]^{\frac{1}{2}} \rho^{\frac{-\mu k_1}{2}}. \tag{5.53}$$

Physical parameters are found using (5.25)

$$\begin{aligned} 2M &= (a\mu - \frac{1}{c_2})\, e_1, \\ \Sigma &= (-\mu - \frac{1}{c_2})\, e_2, \\ Q &= \frac{e_3}{2c_2}. \end{aligned} \tag{5.54}$$

The solution is

$$\begin{aligned} \mu &= -T_2, \\ -\frac{1}{c_2} &= T_1, \end{aligned} \tag{5.55}$$

(5.56)

which gives

$$\frac{2Q}{e_3} = T_1. \tag{5.57}$$

This is the equality case of the inequality (5.36).

CASE 2

$$\Delta = 0 \tag{5.58}$$

Then there is one root to the equation (5.11). Denote it by r_1^d.

$$A\,C^d = (r^d - r_1^d) \tag{5.59}$$

$$A^2 = \frac{(r^d - r_1^d)^2}{C^{2d}}, \qquad B^2 = \frac{r^{2d-2}\,C^2}{(r^d - r_1^d)^2}, \tag{5.60}$$

$$C^d = (r^d - r_1^d)\,\rho^{-\frac{\mu k_1}{2}} \left[\frac{\cos(c_2 + \nu \ln \rho)}{\cos(c_2)}\right]^{k_2} \tag{5.61}$$

Scalar field is given as

$$e^{\alpha_e \phi} = \left[\frac{\cos c_2}{\rho^{\mu}\, \cos(c_2 + \nu \ln \rho)}\right]^{k_1} \tag{5.62}$$

CASE 3

$$\Delta < 0 \tag{5.63}$$

This case is similar to the previous one.

$$e^{\alpha_e \phi} = \left[\frac{\cos c_2}{\rho^{\mu}\, \cos(c_2 + \nu \ln \rho)}\right]^{k_1}, \tag{5.64}$$

$$A^2 = \frac{(r^{2d} - 2b_1 r^d + b_2)}{C^{2d}}, \qquad B^2 = \frac{r^{2d-2}\,C^2}{(r^{2d} - 2b_1 r^d + b_2)}, \tag{5.65}$$

$$C^d = (r^{2d} - 2b_1 r^d + b_2)^{\frac{1}{2}}\,\rho^{-\frac{\mu k_1}{2}} \left[\frac{\cos(c_2 + \nu \ln \rho)}{\cos(c_2)}\right]^{k_2} \tag{5.66}$$

In cases 2 and 3, the relation of physical parameters to integration constants are exactly the same as case 1 type 2, the only difference being the sign of Δ, hence equation(5.49).

Chapter 6

Conclusion

In this study, we considered D-dimensional spherically symmetric space times and obtained the Riemann tensor in compact form. Using this expression, we established certain theorems concerning any spherically symmetric theory of gravitation. As a special case, we considered the Lovelock theory in 5 and 6 dimensions with the LCBR metric, which simplifies most of the expressions. Then we considered the low energy limit of the string theory and obtained black hole solutions carrying mass, electrostatic charge, and dilaton charge. We also generalized an inequality concerning physical parameters of the black hole to D-dimensions.

Chapter 7

Appendix

$$
\begin{aligned}
M^n_n &= \frac{\alpha(D-4)}{2C^2}, \\
M_{in}M^n_j &= \frac{\alpha^2}{4C^4} g_{ij}, \\
M_{in}k^n &= -\frac{\alpha}{2C^2} k_i,
\end{aligned} \tag{7.1}
$$

where

$$
M_{ij} = \alpha(h_{ij} - \frac{1}{2C^2} g_{ij}), \quad \alpha = \frac{(D-3)!}{C^{2(D-3)}}. \tag{7.2}
$$

Using these, we can compute several contractions of S_{ij}, which in turn will be used to compute the contractions of the Riemann tensor. Now we'll define new variables that will simplify the coefficients.

$$
\xi_0 = \frac{\eta_0 \alpha}{C^2}, \; \xi_1 = \frac{\eta_1}{B^2}, \; \xi_2 = \frac{\eta_2 \alpha}{C^2}, \; \xi_3 = \eta_3 \tag{7.3}
$$

$$
\begin{aligned}
S_{ij} &= \frac{C^2}{\alpha} \xi_0 M_{ij} + B^2 \xi_1 k_i k_j + \frac{\xi_3}{2} g_{ij}, \\
S^n_n &= \xi_0 \frac{(D-4)}{2} + \xi_1 + \frac{1}{2}\xi_3 D, \\
S_{in}S^n_j &= \frac{1}{4}(\xi_0^2 + \xi_3^2) g_{ij} + \xi_0 \xi_3 \frac{C^2}{\alpha} M_{ij} + (\xi_1^2 + \xi_1\xi_3 - \xi_0\xi_1) B^2 k_i k_j, \\
S_{in}k^n &= (-\frac{1}{2}\xi_0 + \xi_1 + \frac{1}{2}\xi_3) k_i,
\end{aligned}
$$

$$
\begin{aligned}
S_{in}M_j^n &= \frac{\alpha}{C^2}(\frac{1}{4}\xi_0 g_{ij} + \frac{C^2}{2\alpha}\xi_3 M_{ij} - \frac{1}{2}\xi_1 B^2 k_i k_j), \\
S_i^n h_{jn} &= \frac{1}{2}(\xi_0 + \xi_3) h_{ij}, \\
S^{mn} h_{jn} h_{mi} &= (\xi_0 + \xi_3)\frac{h_{ij}}{2C^2}.
\end{aligned}
\tag{7.4}
$$

Now we can compute the second rank tensors in terms of g_{ij}, M_{ij} and $k_i k_j$

$$
\begin{aligned}
R_i^a R_{ja} &= \beta_1 g_{ij} + \beta_2 M_{ij} + \beta_3 k_i k_j, \\
R_{jaib}R^{ab} &= \beta_4 g_{ij} + \beta_5 M_{ij} + \beta_6 k_i k_j, \\
R_i^{abc} R_{jabc} &= \beta_7 g_{ij} + \beta_8 M_{ij} + \beta_9 k_i k_j, \\
R_{ij} &= f_1 g_{ij} + f_2 M_{ij} + f_3 k_i k_j,
\end{aligned}
\tag{7.5}
$$

where

$$
\begin{aligned}
\beta_1 &= \frac{[(D-3)^2+1]}{2}\xi_0^2 + \xi_1^2 + (D-3)\xi_0\xi_2 + \xi_1\xi_2 + \frac{1}{2}\xi_2^2 + (D-1)(D-4)\xi_0\xi_3 \\
&\quad + (D-4)\xi_0\xi_1 + 2(D-1)\xi_1\xi_3 + (D-1)\xi_2\xi_3 + (D-1)^2\xi_3^2,
\end{aligned}
$$

$$
\begin{aligned}
\beta_2 &= \frac{C^2}{\alpha}[(D-2)(D-4)\xi_0^2 + 2(D-2)\xi_0\xi_1 + 2(D-3)\xi_0\xi_2 + 2\xi_1\xi_2 + \xi_2^2 \\
&\quad + 2(D-1)(D-2)\xi_0\xi_3 + 2(D-1)\xi_2\xi_3,
\end{aligned}
$$

$$
\beta_3 = B^2[-2(D-2)\xi_0\xi_1 + D(D-2)\xi_1^2 + 2(D-1)(D-2)\xi_1\xi_3],
$$

$$
\begin{aligned}
\beta_4 &= \frac{[(D-3)^2+1]}{2}\xi_0^2 - 2\xi_0\xi_1 + (D-1)\xi_1^2 + (D-3)\xi_0\xi_2 + \frac{1}{2}\xi_1\xi_2 + \frac{1}{2}\xi_2^2 \\
&\quad + (\frac{3}{2}D-2)(D-4)\xi_0\xi_3 + (3D-4)\xi_1\xi_3 + \frac{3}{2}(D-2)\xi_2\xi_3 + (D-1)^2\xi_3^2,
\end{aligned}
$$

$$
\begin{aligned}
\beta_5 &= \frac{C^2}{\alpha}[(D-2)(D-4)\xi_0^2 + 2(D-2)\xi_0\xi_1 + 2(D-3)\xi_0\xi_2 + \xi_1\xi_2 + \xi_2^2 \\
&\quad + (D-2)^2\xi_0\xi_3 + (D-2)\xi_2\xi_3],
\end{aligned}
$$

$$
\beta_6 = (D-2)B^2[(D-2)\xi_0\xi_1 + \xi_1\xi_2 + (D-2)\xi_1\xi_3],
$$

$$\beta_7 = (D-2)\xi_0^2 - 2\xi_0\xi_1 + 2\xi_1^2 + 2\xi_0\xi_2 + \frac{1}{(D-3)}\xi_2^2 + 2(D-4)\xi_0\xi_3 + 4\xi_1\xi_3$$
$$+2\xi_2\xi_3 + 2(D-1)\xi_3^2,$$
$$\beta_8 = \frac{C^2}{\alpha}\left[2(D-4)\xi_0^2 + 4\xi_0\xi_1 + 4\xi_0\xi_2 + \frac{2\xi_2^2}{(D-3)} + 4(D-2)\xi_0\xi_3 + 4\xi_2\xi_3\right],$$
$$\beta_9 = 2(D-2)B^2\xi_1(\xi_1 + 2\xi_3) \tag{7.6}$$

and

$$\begin{aligned} f_1 &= \frac{1}{2}(\xi_0(D-4) + \xi_2) + \xi_1 + \xi_3(D-1), \\ f_2 &= \frac{C^2}{\alpha}(\xi_0(D-2) + \xi_2), \\ f_3 &= B^2(D-2)\xi_1. \end{aligned} \tag{7.7}$$

If we insert these in (4.16), we obtain

$$\begin{aligned} &\left[f_1 - \tfrac{1}{2}R + \alpha_0(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} + 8\beta_1 + 8\beta_4 - 4\beta_7 - 4Rf_1)\right] g_{ij} \\ &+ \left[f_2 + \alpha_0(8\beta_2 + 8\beta_5 - 4\beta_8 - 4Rf_2)\right] M_{ij} \\ &+ \left[f_3 + \alpha_0(8\beta_3 + 8\beta_6 - 4\beta_9 - 4Rf_3)\right] k_i k_j = \lambda g_{ij} \end{aligned} \tag{7.8}$$

We can rewrite it as

$$t_1 g_{ij} + t_2 M_{ij} + t_3 k_i k_j = 0 \tag{7.9}$$

So, the field equations will reduce to $t_1 = \lambda,\ t_2 = t_3 = 0$, where these coefficients are given as

$$
\begin{aligned}
t_1 &= -\frac{1}{2}(D-4)(D-2)\xi_0 - (D-2)\xi_1 - \frac{1}{2}(D-3)\xi_2 - \frac{1}{2}(D-1)(D-2)\xi_3 \\
&\quad +\alpha_0(D-3)(D-4)[(D-3)(D-6)\xi_0^2 + 4(D-3)\xi_0\xi_1 + 2(D-6)\xi_0\xi_2 \\
&\quad +4\xi_1\xi_2 + \frac{(D-4)(D-5)}{(D-3)^2}\xi_2^2 + 2(D-4)(D-2)\xi_0\xi_3 + 4(D-2)\xi_1\xi_3 \\
&\quad +2(D-3)\xi_2\xi_3 + (D-1)(D-2)\xi_3^2], \qquad (7.10)
\end{aligned}
$$

$$
\begin{aligned}
t_2 &= \frac{C^2}{\alpha}[(D-2)\xi_0 + \xi_2] - 4\frac{C^2}{\alpha}\alpha_0(D-3)(D-4)\left[(D-4)\xi_0^2 + \frac{(2D-9)}{(D-3)}\xi_0\xi_2\right. \\
&\quad \left. +2\xi_0\xi_1 + \frac{2}{(D-3)}\xi_1\xi_2 + \frac{(D-5)}{(D-3)^2}\xi_2^2 + (D-2)\xi_0\xi_3 + \xi_2\xi_3\right], \qquad (7.11)
\end{aligned}
$$

$$
\begin{aligned}
t_3 &= -4\alpha_0 B^2(D-2)(D-4)\xi_1[\xi_0(D-3) + \xi_2 + (D-3)\xi_3] \\
&\quad +B^2(D-2)\xi_1. \qquad (7.12)
\end{aligned}
$$

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